Finding Eigenvectors of a Matrix

Recall: Let A be a $n \times n$ matrix. We call a vector $\boldsymbol{x}$ in $\mathbb{R}^n$ an *eigenvector* of A with corresponding *eigenvalue* λ (a scalar) if

$$A\boldsymbol{x} = \lambda \boldsymbol{x}, \qquad \boldsymbol{x} \neq \mathbf{0} \tag{1}$$

Example: Consider a 3×3 matrix A and suppose that

$$A \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \tag{2}$$

Find an eigenvector of A and the corresponding eigenvalue. Find another eigenvector of A.

$$A \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \underline{3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 3$.

$$A \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = A \left(2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = 2A \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -6 \end{bmatrix} = \underline{3} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

The vector $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 3$.

Theorem 1: Let A be a $n \times n$ matrix. Then a vector $\boldsymbol{x}$ in $\mathbb{R}^n$ is an *eigenvector* of A with corresponding *eigenvalue* λ (a scalar) if and only if

$$(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}, \qquad \boldsymbol{x} \neq \boldsymbol{0} \tag{3}$$

Proof Idea: $(A - \lambda I)\vec{x} = \vec{0} \Leftrightarrow A\vec{x} - \lambda I\vec{x} = 0 \Leftrightarrow A\vec{x} = \lambda\vec{x}$

Example: Consider a 3×3 matrix A and suppose that

$$(A - 2I)\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{4}$$

Find an eigenvector of A and the corresponding eigenvalue. Find another eigenvector of A.

$(A - \lambda I)\vec{x} = \vec{0}$ where $\lambda = 2$ $\vec{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

By theorem 1, $\vec{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 2$.

$\begin{bmatrix}1\\1\\1\end{bmatrix}$ in null $(A - 2I)$.

Recall that null$(A - 2I)$ is a subspace.

Thus $\begin{bmatrix}2\\2\\2\end{bmatrix}$ in null $(A - 2I)$.

By theorem 1, $\begin{bmatrix}2\\2\\2\end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 2$.